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On Deterministic Indexed Languages

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A new type of acceptor is introduced for the class of indexed languages. The class of languages recognized by the deterministic version of this acceptor, called deterministic indexed languages, is closed under complemention. The class of deterministic context-free languages is properly contained in the class of deterministic indexed languages, which itself is properly contained in the class of indexed languages.

1. INTRODUCTION

Aho (1968) extended the class of context-free languages to the class of indexed languages. The class of indexed languages has properties which are analogous to those of the class of context-free languages. In the area of context-free languages one is particularly interested in deterministic context-free languages, that is, languages which are accepted by deterministic pushdown automata, because this subset of the context-free languages is of interest with regard to programming languages.

A new type of recognizer will be introduced for indexed languages, which is a natural model for syntactic analyzers of these languages. A deterministic version of these recognizers singles out a subset of the indexed languages, the deterministic indexed languages (DIL's).

As in the case of context-free languages, the DIL's are a proper subset of the indexed languages because, as will be shown, the complement of a DIL is likewise a DIL. In addition it will be shown that DIL's properly contain the deterministic context-free languages and that there are DIL's which are not context-free.

2. INDEXED PUSHDOWN AUTOMATA AND DETERMINISTIC INDEXED LANGUAGES

In this section indexed pushdown automata (IPDA's) will be defined and it will be shown that these automata accept the class of indexed languages. IPDA's are an extension of ordinary pushdown automata. Indexed pushdown automata are constructed by attaching a list of indices, called the index list, to each symbol

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of the pushdown list. In addition the transition function is modified so that each transition also depends on the first element of the index list.

DEFINITION 2.1. An *indexed pushdown automaton* (*IPDA*) is a 9-tuple $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$, where

(1) Z is a finite set of *states*,

(2) X is a finite set of *input symbols*,

(3) Γ_1 is a finite set of *pushdown list symbols*,

(4) Γ_2 is a finite set of *indices*,

(5) $z_0 \in Z$ is the *initial state*,

(6) $A_0 \in \Gamma_1$ is the *initial pushdown list symbol*,

(7) $g_0 \in \Gamma_2 \cup \{e\}$ is the *initial index*,

(8) $F \subseteq Z$ is the set of *final states*,

(9) δ is the *transition function*: δ is a mapping from $Z \times (X \cup \{e\}) \times (\Gamma_1 \times (\Gamma_2 \cup \{e\}))$ to the finite subsets of $Z \times (\Gamma_1 \times \Gamma_2^*)^*$.
(e denotes the empty word.)

A triple (z, w, θ) with $z \in Z$, $w \in X^*$, and $\theta \in (\Gamma_1 \times \Gamma_2^*)^*$ is called a *configuration of* K. A binary relation $\vdash$ on the set of configurations of K is defined as follows:

$$(z, xw, (A, g\gamma)\theta) \vdash (z', w, (B_1, \beta_1\gamma) \cdots (B_r, \beta_r\gamma)\theta) \text{ iff}$$

$$(z', (B_1, \beta_1) \cdots (B_r, \beta_r)) \in \delta(z, x, (A, g)) \text{ for } z, z' \in Z,$$

$$x \in X \cup \{e\},\ w \in X^*,\ A, B_1, \ldots, B_r \in \Gamma_1,\ g \in \Gamma_2 \cup \{e\},$$

$$\gamma, \beta_1, \ldots, \beta_r \in \Gamma_2^*,\ r \geqslant 0, \text{ and } \theta \in (\Gamma_1 \times \Gamma_2^*)^*.$$

If $x = e$, we say that the second configuration is obtained from the first by an *e-move*.

$\vdash^+$ is the transitive, $\vdash^*$ the reflexive and transitive closure of $\vdash$, and $\vdash^n$ is the n-fold product of $\vdash$ with $n \geqslant 0$.

A word $w \in X^*$ is *accepted by* K if $(z_0, w, (A_0, g_0)) \vdash^* (z, e, \theta)$ for some $z \in F$.

The *language accepted by* K, denoted $L(K)$, is the set of words accepted by K.

The *language accepted by* K *with an empty pushdown list*, is the set $L_e(K) = \{w \mid (z_0, w, (A_0, g_0)) \vdash^* (z, e, e) \text{ with } z \in Z\}$.

Two IPDA's K and K' are called *equivalent*, if $L(K) = L(K')$ holds.

EXAMPLE 2.1. Set $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$ with $Z = \{z_0, z_1, z_2, z_3\}$, $X = \{a, b, c\}$, $\Gamma_1 = \{A, A_0\}$, $\Gamma_2 = \{g, g_0\}$, $F = \{z_0\}$, and δ is defined as follows:

$$\begin{aligned}
\delta(z_0, a, (A_0, e)) &= \{(z_1, (A_0, g))\} \\
\delta(z_1, a, (A_0, e)) &= \{(z_1, (A_0, g))\} \\
\delta(z_1, b, (A_0, g)) &= \{(z_2, (A, e)(A_0, g))\} \\
\delta(z_2, b, (A, g)) &= \{(z_2, (A, e))\} \\
\delta(z_2, e, (A, g_0)) &= \{(z_3, e)\} \\
\delta(z_3, c, (A_0, g)) &= \{(z_3, (A_0, e))\} \\
\delta(z_3, e, (A_0, g_0)) &= \{(z_0, e)\}.
\end{aligned}$$

The word $w = a^3b^3c^3$, for example, is accepted by K with the following sequence of moves:

$$\begin{aligned}
(z_0, w, (A_0, g_0)) &\vdash (z_1, a^2b^3c^3, (A_0, gg_0)) \\
&\overset{2}{\vdash} (z_1, b^3c^3, (A_0, g^3g_0)) \\
&\vdash (z_2, b^2c^3, (A, g^2g_0)(A_0, g^3g_0)) \\
&\overset{2}{\vdash} (z_2, c^3, (A, g_0)(A_0, g^3g_0)) \\
&\vdash (z_3, c^3, (A_0, g^3g_0)) \\
&\overset{3}{\vdash} (z_3, e, (A_0, g_0)) \\
&\vdash (z_0, e, e).
\end{aligned}$$

K accepts the language $L(K) = \{a^nb^nc^n \mid n \geqslant 0\}$.

In this example a significant property of IPDA's is illustrated, namely that information can be passed from one index list to the index list above it.

As in the case of pushdown automata it can be shown that the languages accepted by IPDA's with an empty pushdown list are exactly the languages accepted by IPDA's with final states.

THEOREM 2.1. (i) *Let $L = L(K)$ for an IPDA K. Then there is an IPDA K' with $L = L_e(K')$.*

(ii) *Let $L = L_e(K)$ for an IPDA K. Then there is an IPDA K' with $L = L(K')$.*

The proof of the theorem is analogous to the corresponding proof for pushdown automata and is therefore omitted.

The following theorem states that the languages accepted by IPDA's are exactly the indexed languages. For the definition of indexed grammars and indexed languages see Aho (1968).

THEOREM 2.2. *A language L is accepted by an IPDA iff L is an indexed language.*

Proof. Let L be an indexed language. There is an indexed grammar $G =$

(N, T, I, P, S) in reduced form (cf. Aho, 1968), which generates L. Each index production in each index $f \in I$ is of the form $A \to B$, where both A and B are in N. Each production in P is of one of the forms

(1) $A \to BC$, or

(2) $A \to Bf$, or

(3) $A \to a$

with $A, B, C \in N, f \in I$, and $a \in T \cup \{e\}$.

Set $K = (Z, T, N, I, \delta, z_0, S, e, \varnothing)$ with $Z = \{z_0\}$, and δ is defined as follows:

For each index production $A \to B$ in each index $f \in I$ set $(z_0, (B, e)) \in \delta(z_0, e, (A, f))$.

For each production in P of the form

(1) $A \to BC$ set $(z_0, (B, e)(C, e)) \in \delta(z_0, e, (A, e))$,

(2) $A \to Bf$ set $(z_0, (B, f)) \in \delta(z_0, e, (A, e))$,

(3) $A \to a$ set $(z_0, e) \in \delta(z_0, a, (A, e))$.

Obviously $L = L_e(K)$ holds. Now for the converse, let $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$ be an IPDA with $L_e(K) = L$. Define an indexed grammar $G = (N, X, I, P, S)$ in the following way:

$$N = (\bar{Z} \times \Gamma_1 \times Z) \cup \{S\}$$

with

$$\bar{Z} = Z \cup \{z_g \mid z \in Z, g \in \Gamma_2\}, I = \bar{\Gamma}_2 = \{\bar{g} \mid g \in \Gamma_2\}.$$

For all $z \in Z$, P contains the productions $S \to [z_0, A_0, z]\, \bar{g}_0$. (It is assumed that $\bar{e} = e$.)

For all $z \in Z$, $x \in X \cup \{e\}$ and $A \in \Gamma_1$: If $(z', (B_1, \beta_1) \cdots (B_n, \beta_n)) \in \delta(z, x, (A, e))$ with $n \geqslant 0$, then P contains the productions

$$[z, A, z_n] \to x[z', B_1, z_1]\, \bar{\beta}_1 [z_1, B_2, z_2]\, \bar{\beta}_2 \cdots [z_{n-1}, B_n, z_n]\, \bar{\beta}_n$$

for every sequence $z_1, ..., z_n$ in Z. (If $\beta = g_1 \cdots g_k$, then $\bar{\beta} = \bar{g}_1 \cdots \bar{g}_k$.) If $(z', e) \in \delta(z, x, (A, e))$ then P contains the production $[z, A, z'] \to x$.

For all $z \in Z$, $x \in X \cup \{e\}$, $A \in \Gamma_1$ and $g \in \Gamma_2$: If $(z', (B_1, \beta_1) \cdots (B_n, \beta_n)) \in \delta(z, x, (A, g))$ with $n \geqslant 0$, then P contains the productions

$$[z_g, A, z_n] \to x[z', B_1, z_1]\, \bar{\beta}_1 [z_1, B_2, z_2]\, \bar{\beta}_2 \cdots [z_{n-1}, B_n, z_n]\, \bar{\beta}_n$$

and $\bar{g}$ contains the index production $[z, A, z_n] \to [z_g, A, z_n]$ for every sequence $z_1, ..., z_n \in Z$. If $(z', e) \in \delta(z, x, (A, g))$ then $\bar{g}$ contains the index production $[z, A, z'] \to x$.

It is easy to show that $L = L(G)$.

IPDA's which can make at most one move in any configuration are called deterministic. This concept is made precise in the following definition.

DEFINITION 2.2. An IPDA $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$ is called *deterministic IPDA* (*d-IPDA*) if $|\delta(z, x, (A, g))| \leqslant 1$ holds for all $z \in Z$, $x \in X \cup \{e\}$, $A \in \Gamma_1$, $g \in \Gamma_2 \cup \{e\}$, and the following conditions are satisfied for all $z \in Z$ and $A \in \Gamma_1$:

(1) If $\delta(z, x, (A, g)) \neq \varnothing$ for some $x \in X$ and $g \in \Gamma_2$, then $\delta(z, x', (A, g')) = \varnothing$ for $x' \in \{x, e\}$ and $g' \in \{g, e\}$ with $|x'g'| \leqslant 1$.

(2) If $\delta(z, x, (A, e)) \neq \varnothing$ for some $x \in X$, then $\delta(z, x', (A, g)) = \varnothing$ for $x' \in \{x, e\}$ and for all $g \in \Gamma_2$ and $\delta(z, e, (A, e)) = \varnothing$.

(3) If $\delta(z, e, (A, g)) \neq \varnothing$ for some $g \in \Gamma_2$, then $\delta(z, x, (A, g')) = \varnothing$ for $g' \in \{g, e\}$ and for all $x \in X$ and $\delta(z, e, (A, e)) = \varnothing$.

(4) If $\delta(z, e, (A, e)) \neq \varnothing$, then $\delta(z, x', (A, g')) = \varnothing$ for all $x' \in X \cup \{e\}$, $g' \in \Gamma_2 \cup \{e\}$ with $|x'g'| \geqslant 1$. ($|w|$ denotes the length of w.)

A language which is accepted by a d-IPDA is called a *deterministic indexed language* (*DIL*).

Convention. Since $\delta(z, x, (A, g))$ contains at most one element for a d-IPDA, $\delta(z, x, (A, g)) = (z', \theta)$ will be written instead of $\delta(z, x, (A, g)) = \{(z', \theta)\}$.

Remark. The language $\{a^n b^n c^n \mid n \geqslant 1\}$ presented in Example 1 is a DIL, but is not context-free, as is well known. Hence the class of context-free languages is a proper subset of the class of indexed languages.

The fact that deterministic context-free languages are closed under complementation but context-free languages in general are not, permits the proof that the class of deterministic context-free languages is a proper subset of the class of context-free languages. In the next section the same relationship between DIL's and indexed languages in general will be shown.

3. CLOSURE OF DETERMINISTIC INDEXED LANGUAGES UNDER COMPLEMENTATION

In the proof of closure of deterministic context-free languages under complementation, a new deterministic pushdown automaton is constructed from a continuing deterministic automaton which accepts exactly those strings not accepted by the continuing deterministic pushdown automaton. In the construction it is necessary to determine combinations of states and pushdown list symbols with a certain property (looping) and modify the transition function of these combinations (cf. Aho and Ullman, 1972). However, in the case of d-IPDA's an arbitrarily long index list must be taken into consideration and so an analogous construction is not possible. It is, on the other hand, possible to construct the d-IPDA which accepts the opposite input strings by means of a series of appropriate transformations on the d-IPDA.

It will be shown next that for every d-IPDA there is an equivalent one whose transition function is always dependent on the first symbol in the index list.

DEFINITION 3.1. A d-IPDA $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$ is called *d_0-IPDA* if $g_0 \neq e$ and $\delta(z, x, (A, e)) = \varnothing$ holds for all $z \in Z$, $x \in X \cup \{e\}$, and $A \in \Gamma_1$.

THEOREM 3.1. *For each d-IPDA K there exists an equivalent d_0-IPDA K'.*

Proof. Let $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$ and set $K' = (Z', X, \Gamma_1, \Gamma_2, \delta', z_0', A_0, \#, F)$ with $Z' = Z \cup \{z_0'\}$ and $\Gamma_2' = \Gamma_2 \cup \{\#\}$. δ' is defined in the following way:

(1) $\delta'(z_0', e, (A_0, \#)) = (z_0, (A_0, g_0\#))$.

(2) For all $z \in Z$, $x \in X \cup \{e\}$, $A \in \Gamma_1$, $g \in \Gamma_2$ set $\delta'(z, x, (A, g)) = \delta(z, x, (A, g))$.

(3) For all $z \in Z$, $x \in X \cup \{e\}$, $A \in \Gamma_1$ with $\delta(z, x, (A, e)) = (z', (B_1, \beta_1) \cdots (B_r, \beta_r))$ set $\delta'(z, x, (A, g)) = (z', (B_1, \beta_1 g) \cdots (B_r, \beta_r g))$ for all $g \in \Gamma_2'$. K' is a d_0-IPDA and we have $L(K) = L(K')$.

For the following transformation it is convenient to express the transition function in as simple a form as possible. For this reason the following normal form is defined.

DEFINITION 3.2. A d_0-IPDA $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$ is called *normalized*, if the following is satisfied: For all $z \in Z$, $x \in X \cup \{e\}$, $A \in \Gamma_1$, and $g \in \Gamma_2$ the set $\delta(z, x, (A, g))$ only contains pairs of the form

(1) $(z', (B, e))$ or

(2) $(z', (B, fg))$ or

(3) $(z', (A, g)(B, g))$ or

(4) (z', e)

with $z' \in Z$, $B \in \Gamma_1$, and $f \in \Gamma_2 \cup \{e\}$. Furthermore, if $x \neq e$, $\delta(z, x, (A, g))$ only contains pairs of the form (1).

THEOREM 3.1. *For each d_0-IPDA K there exists an equivalent normalized d_0-IPDA.*

Proof. Assume $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$ is not normalized. Define $K' = (Z, X, \Gamma_1', \Gamma_2', \delta', z_0, A_0, \#, F)$ with $\Gamma = \Gamma_2 \cup \{\#\}$ and $\Gamma_1' = \Gamma_1 \cup (\Gamma_1 \times \bigcup_{k=1}^{m} \Gamma_2{}^k)$, where m is the maximal length of the index lists β_i appearing in the images of the transition funktion δ.

δ' is defined by $\delta'(z_0, e, (A_0, \#)) = (z_0, (A_0, g_0\#))$ and for all $z \in Z$, $x \in X \cup \{e\}$, $A \in \Gamma_1$, $g \in \Gamma_2$

$$\delta'(z, x, (A, g)) = \delta(z, x, (A, g))$$

and for all $z \in Z$, $A \in \Gamma_1$, $\gamma \in \Gamma_2^*$ with $|\gamma| \leqslant m - 1$, $g \in \Gamma_2'$, $f \in \Gamma_2$:

$$\begin{aligned} \delta'(z, e, ((A, \gamma f), g)) &= (z, ((A, \gamma), fg)) && \text{if } \ |\gamma| > 0 \\ &= (z, (A, fg)) && \text{if } \gamma = e. \end{aligned}$$

Clearly, we have $L(K) = L(K')$, and K' is a d_0-IPDA, which is not normalized.

Select a triple $(z, x, (A, g))$, such that $\delta'(z, x, (A, g))$ does not satisfy the conditions of Definition 3.2. Obviously $A \in \Gamma_1$ and $g \in \Gamma_2$ holds. Let

$$\delta'(z, x, (A, g)) = (\tilde{z}, (B_1, \beta_1)(B_2, \beta_2) \cdots (B_r, \beta_r)).$$

(If $x \neq e$, this pair is not of the form (1), if $x = e$, this pair is of none of the forms (1)–(4).)

Now define an IPDA $\tilde{K} = (\tilde{Z}, X, \Gamma_1', \Gamma_2', \tilde{\delta}, z_0, A_0, \#, F)$ with $\tilde{Z} = Z \cup \{z_1, ..., z_{r+1}\}$ ($z_1, ..., z_{r+1}$ are new states). $\tilde{\delta}$ is defined by the following cases:

(1) For all $(z', x', (A', g')) \in Z \times (X \cup \{e\}) \times (\Gamma_1' \times (\Gamma_2 \cup \{e\}))$ with $(z', x', (A', g')) \neq (z, x, (A, g))$ let

$$\tilde{\delta}(z', x', (A', g')) = \delta'(z', x', (A', g')).$$

(2) $\tilde{\delta}(z, x, (A, g)) = (z_1, (A, e))$ and for all $g' \in \Gamma_2'$ let

$$\tilde{\delta}(z_{i-1}, e, (A, g')) = (z_i, (A, g')(C_{r-(i-2)}, g'))$$

with $i \in [2\colon r + 1]$, where

$$\begin{aligned} C_j &= (B_j, \beta_j) && \text{if } \ \beta_j \neq e \\ &= B_j && \text{otherwise.} \end{aligned}$$

(3) $\tilde{\delta}(z_{r+1}, e, (A, g')) = (\tilde{Z}, e)$. $\tilde{K}$ is a d_0-IPDA with $L(\tilde{K}) = L(K')$.

If $\tilde{K}$ is not normalized repeat the construction with $\tilde{K}$ in place of K'. After a finite number of steps we obtain a normalized d_0-IPDA $\bar{K}$ with $L(K) = L(\bar{K})$.

COROLLARY 3.1. *For each d-IPDA there exists an equivalent normalized d_0-IPDA.*

The next transformation modifies a d-IPDA such that, if the automaton is in a final state all subsequent states obtained by e-moves alone are final states. The same transformation is used in Rosenkrantz and Hunt (1978).

DEFINITION 3.3. An IPDA $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$ is called *F-preserving*, if for all $z_1, z_2 \in Z$, $\theta_1, \theta_2 \in (\Gamma_1 \times \Gamma_2^*)^*$ the following is satisfied:

$$(z_1, e, \theta_1) \overset{*}{\vdash} (z_2, e, \theta_2) \quad \text{with} \quad z_1 \in F \quad \text{implies} \quad z_2 \in F.$$

THEOREM 3.2. *For each (normalized) d_0-IPDA K there exists an equivalent F-preserving (normalized) d_0-IPDA.*

Proof. Let $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$. Form

$K' = (Z', X, \Gamma_1, \Gamma_2, \delta', z_0, A_0, g_0, F)$ with $Z' = Z \cup \hat{Z}$ and $F' = F \cup \hat{Z}$,

where $\hat{Z} = \{\hat{z} \mid z \in Z\}$ is a duplicate of Z, and define δ' by:
For all $z_1 \in Z$, $x \in X \cup \{e\}$, $A \in \Gamma_1$, $g \in \Gamma_2$ let

$$\begin{aligned} \delta'(z_1, x, (A, g)) &= \delta(z_1, x, (A, g)) \quad \text{if} \quad x \neq e \quad \text{or} \quad z_1 \notin F \\ &= \{(\hat{z}_2, \theta) | (z_2, \theta) \in \delta(z_1, x, (A, g))\} \quad \text{if} \quad x = e \quad \text{and} \quad z_1 \in F \end{aligned}$$

and

$$\begin{aligned} \delta'(\hat{z}_1, x, (A, g)) &= \{(z_2, \theta) | (z_2, \theta) \in \delta(z_1, x, (A, g)) \quad \text{if} \quad x \in X \\ &= \{(\hat{z}_2, \theta) | (z_2, \theta) \in \delta(z_1, x, (A, g)) \quad \text{if} \quad x = e. \end{aligned}$$

K' is a F-preserving (normalized) d_0-IPDA.
It is easy to show that $L(K) = L(K')$.

The following transformation modifies a d-IPDA such that after a symbol has been pushed on the pushdown list by an e-move then the pushdown list cannot be shortened by subsequent e-moves.

DEFINITION 3.4. An IPDA $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$ is called *e-monotone* if

$$(z_0, v, (A_0, g_0)) \overset{*}{\vdash} (z, e, \theta) \vdash (z', e, \theta') \overset{*}{\vdash} (z'', e, \theta'')$$

with $|\theta| < |\theta'|$ implies $|\theta'| \leqslant |\theta''|$ for all $v \in X^*$, $z, z', z'' \in Z$, $\theta, \theta', \theta'' \in (\Gamma_1 \times \Gamma_2^*)^*$.

Remark. Let K be e-monotone. Then the lengths of the pushdown lists of the configurations following (z', e, θ') are monotone increasing.

The transformation of a d-IPDA to e-monotone form requires, first of all, some conceptual preparation. Let $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$ be a normalized d_0-IPDA, $\mathscr{M}_K = Z \times \Gamma_1 \times Z$ and $\psi_K: \Gamma_2^* \to \mathscr{P}(\mathscr{M}_K)$ be the function defined by $\psi_K(\gamma) = \{(z_1, A, z_2) | (z_1, e, (A, \gamma)) \overset{*}{\vdash}_K (z_2, e, e)\}$. ($\mathscr{P}(\mathscr{M}_K)$ denotes the power set of $\mathscr{M}_K$.) Then $\psi_K(e) = \varnothing$, and since K is deterministic, (z_1, A, z_2),

$(z_1, A, z_3) \in \psi_K(\gamma)$ implies $z_2 = z_3$. (Given (z_1, A, z_2) and γ, it is decidable whether $(z_1, A, z_2) \in \psi_K(\gamma)$.)

ψ_K assigns to each index list a set ofttriples (z_1, A, z_2) with the following property: If the automaton is in state z_1 and the pushdown symbol A with the attached index list γ is on the top of the pushdown list, then the automaton passes to state z_2 by a sequence of e-moves and (A, γ) is deleted.

LEMMA 3.1. *If* $\psi_K(\gamma) = \psi_K(\gamma')$ *then* $\psi_K(\alpha\gamma) = \psi_K(\alpha\gamma')$ *for all* $\gamma, \gamma', \alpha \in \Gamma_2^*$.

Proof. Let $|\alpha| > 0$ and $(z_1, A, z_2) \in \psi_K(\alpha\gamma)$. If $(z_1, e, (A, \alpha\gamma)) \vdash^1 (z_2, e, e)$ then $(z_1, e, (A, \alpha\gamma')) \vdash^1 (z_2, e, e)$, hence $(z_1, A, z_2) \in \psi_K(\alpha\gamma')$, too.

Assume, for all $\alpha \in \Gamma_2^+$, $(z_1, A, z_2) \in \psi_K(\alpha\gamma)$ with $(z_1, e, (A, \alpha\gamma)) \vdash^m (z_2, e, e)$, where $m \leqslant n$, then $(z_1, A, z_2) \in \psi_K(\alpha\gamma')$. Now let $\alpha = g\alpha'$, $g \in \Gamma_2$, and $(z_1, A, z_2) \in \psi_K(\alpha\gamma)$ with $(z_1, e, (A, \alpha\gamma)) \vdash^{n+1} (z_2, e, e)$. Consider the following three cases:

(1) $(z_1, e, (A, g\alpha'\gamma)) \vdash (z', e, (A, \alpha'\gamma)) \vdash^n (z_2, e, e)$. Then $(z_1, A, z_2) \in \psi_K(\alpha\gamma')$.

(2) $(z_1, e, (A, \alpha\gamma)) \vdash (z', e, (B, f\alpha\gamma)) \vdash^n (z_2, e, e)$ with $f \in \Gamma_2 \cup \{e\}$. Then $(z_1, A, z_2) \in \psi_K(\alpha\gamma')$.

(3) $(z_1, e, (A, \alpha\gamma)) \vdash (z', e, (A, \alpha\gamma)(B, \alpha\gamma)) \vdash^n (z_2, e, e)$. Select n_1 minimal with

$$\begin{aligned} (z_1, e, (A, \alpha\gamma)) &\vdash (z', e, (A, \alpha\gamma)(B, \alpha\gamma)) \\ &\overset{n_1}{\vdash} (z'', e, (B, \alpha\gamma)) \overset{n_2}{\vdash} (z_2, e, e) \qquad (n_1, n_2 < n) \end{aligned}$$

A corresponding sequence of moves is possible, where γ is substituted by γ'. Hence $(z_1, A, z_2) \in \psi_K(\alpha\gamma')$. Thus we have $\psi_K(\alpha\gamma) \subseteq \psi_K(\alpha\gamma')$. By symmetry the converse inclusion holds as well.

Set $\psi_K(\Gamma_2^*) = \mathscr{P}_1(\mathscr{M}_K)$ and $\mathscr{P}_2(\mathscr{M}_K) = \mathscr{P}(\mathscr{M}_K)/\mathscr{P}_1(\mathscr{M}_K)$. The function $\phi_K: \Gamma_2 \times \mathscr{P}(\mathscr{M}_K) \to \mathscr{P}(\mathscr{M}_K)$ is defined by:

If $M \in \mathscr{P}_1(\mathscr{M}_K)$, then $\phi_K(g, M) = \psi_K(g\gamma)$, where $M = \psi_K(\gamma)$, if $M \in \mathscr{P}_2(\mathscr{M}_K)$, then $\phi_K(g, M) = \varnothing$.

From Lemma 3.1. follows that ϕ_K is well defined.

THEOREM 3.3. *For each d-IPDA K there exists an equivalent normalized F-preserving d_0-IPDA, which is e-monotone.*

In the following proof we will construct a d_0-IPDA K' which simulates K, with one exception. All sequences of e-moves, which first increase and then decrease the length of the pushdown list, will be replaced by a single move. In order to make this replacement it must be decided for each sequence of moves of the form

$$(z_0, w, (A_0, g_0)) \overset{*}{\vdash} (z_1, v, (A, \gamma)\theta) \vdash (z_2, v, (A, \gamma)(B, \gamma)\theta)$$

whether one can reach a configuration (z_3, e, e) starting from $(z_2, e, (A, \gamma))$, i.e., whether there exists $z_3 \in Z$ with $(z_2, A, z_3) \in \psi_K(\gamma)$. To this end the pushdown symbols and the indices are extended by a second component containing subsets of $\mathscr{M}_K$. The transition function of K will be modified in the following way: If K reaches the configuration $(z_1, v, (A, \gamma)\theta)$, K' reaches a corresponding configuration where the topmost pushdown list symbol is the pair $(A, \psi_K(\gamma))$. This modification is possible for the following reason: If an index g is written ahead of the index list γ, then the second component of the topmost pushdown list symbol has to be $\psi_K(g\gamma)$. But this is just $\phi_K(g, \psi_K(\gamma))$ (cf. Lemma 3.1.).

Proof. Assume that $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$ is a normalized and F-preserving d_0-IPDA. Set $K' = (Z, X, \Gamma_1', \Gamma_2', \delta', z_0, A_0', g_0', F)$, where $\Gamma_1' = \Gamma_1 \times \mathscr{P}(\mathscr{M}_K)$, $\Gamma_2' = \Gamma_2 \times \mathscr{P}(\mathscr{M}_K)$, $A_0' = (A_0, M_0)$ with $M_0 = \psi_K(g_0)$, $g_0' = (g_0, \varnothing)$, and for all $z_1 \in Z$, $x \in X \cup \{e\}$, $A \in \Gamma_1$, $M, M' \in \mathscr{P}(\mathscr{M}_K)$, $g \in \Gamma_2$ δ' is defined by the following four cases:

(1) If $\delta(z_1, x, (A, g)) = (z_2, (B, e))$ then

$$\delta'(z_1, x, ((A, M), (g, M'))) = (z_2, ((B, M'), e)).$$

(2) If $\delta(z_1, e, (A, g)) = (z_2, (B, g))$ then

$$\delta'(z_1, e, ((A, M), (g, M'))) = (z_2, ((B, M), (g, M'))).$$

If $\delta(z_1, e, (A, g)) = (z_2, (B, fg))$ with $f \in \Gamma_2$ then

$$\delta'(z_1, e, ((A, M), (g, M'))) = (z_2, ((B, M''), (f, M)(g, M'))),$$

where $M'' = \phi_K(f, M)$.

(3) If $\delta(z_1, e, (A, g)) = (z_2, (A, g)(B, g))$ then

$$\begin{aligned}
&\delta'(z_1, e, ((A, M), (g, M'))) \\
&\quad = (z_2, ((A, M), (g, M'))((B, M), (g, M'))) \quad \text{if for all } z_3 \in Z\text{: } (z_2, A, z_3) \notin M \\
&\quad = (z_3, ((B, M), (g, M'))) \quad \text{if there is a } z_3 \in Z \text{ with } (z_2, A, z_3) \in M.
\end{aligned}$$

(4) If $\delta(z_1, e, (A, g)) = (z_2, e)$ then

$$\delta'(z_1, e, ((A, M), (g, M'))) = (z_2, e).$$

K' is a normalized, F-preserving d_0-IPDA.

Let Φ: $(\Gamma_1 \times \Gamma_2^*)^* \to (\Gamma_2' \times \Gamma_2'^*)^*$ be the homomorphism defined by

$$\Phi(A, f_1 \cdots f_r) = ((A, \psi_K(f_1 \cdots f_r)), (f_1, \psi_K(f_2 \cdots f_r)) \cdots (f_r, \psi_K(e)))$$

for all $A \in \Gamma_1$ and $f_1 \cdots f_r \in \Gamma_2$ with $r \geqslant 0$. With the aid of Φ we can state the following two assertions:

(a) If $(z, w, \theta) \vdash_K (z', w', \theta')$ with $|\theta'| \leqslant |\theta|$ then

$$(z, w, \Phi(\theta)) \vdash_{K'} (z', w', \Phi(\theta')).$$

This holds, for $|\theta'| \leqslant |\theta|$ implies that the move $(z, w, \theta) \vdash_K (z', w', \theta')$ is determined by one of the cases (1), (2), or (4). Then the corresponding move in K' is possible.

(b) If $(z, w, (A, \gamma)\theta) \vdash_K (z', w', (A, \gamma)(B, \gamma)\theta)$ with $(z', A, z'') \notin \psi_K(\gamma)$ for all $z'' \in Z$ then

$$(z, w, \Phi((A, \gamma)\theta)) \vdash_{K'} (z', w', \Phi((A, \gamma)(B, \gamma)\theta)).$$

If $(z, w, (A, \gamma)\theta) \vdash_K (z', w', (A, \gamma)(B, \gamma)\theta)$ with $(z', A, z'') \in \psi_K(\gamma)$ for a $z'' \in Z$ then

$$(z, w, \Phi((A, \gamma)\theta)) \vdash_{K'} (z'', w', \Phi((B, \gamma)\theta)).$$

First $L(K) \subseteq L(K')$ will be shown.

For that purpose it will be proved by induction on n:

If $(z, v, \theta) \vdash_K^n (z', e, \theta')$ with $z' \in F$ then

$$(z, v, \Phi(\theta)) \vdash_{K'}^* (z'', e, \theta'') \qquad \text{with} \quad z'' \in F.$$

This holds for $n = 0$. Assume the assertion holds for all $k \leqslant n$. Let $(z, v, \theta) \vdash_K (z_1, v_1, \theta_1) \vdash_K^n (z', e, \theta')$. If the first move is determined by (1), (2), or (4) then $(z, v, \Phi(\theta)) \vdash_{K'} (z_1, v_1, \Phi(\theta_1))$ and with the induction hypothesis $(z_1, v_1, \Phi(\theta_1)) \vdash_{K'}^* (z'', e, \theta'')$ with $z'' \in F$ follows. If the first move is determined by (3) then

$$(z, v, (A, \gamma)\, \theta_2) \vdash_K (z_1, v, (A, \gamma)(B, \gamma)\, \theta_2) \vdash_K^n (z', e, \theta').$$

Assume $(z_1, A, \hat{z}) \notin \psi_K(\gamma)$ for all $\hat{z} \in Z$. Then

$$(z, v, \Phi((A, \gamma)\, \theta_2)) \vdash_{K'} (z_1, v, \Phi((A, \gamma)(B, \gamma)\, \theta_2)) \vdash_{K'}^* (z'', e, \theta'')$$

with $z'' \in F$. Suppose there exists a $\hat{z} \in Z$ with $(z_1, A, \hat{z}) \in \psi_K(\gamma)$, then there is a uniquely determined m with $(z_1, e, (A, \gamma)) \vdash_K^m (\hat{z}, e, e)$. If $m \leqslant n$ then

$$(z, v, (A, \gamma)\, \theta_2) \vdash_K (z_1, v, (A, \gamma)(B, \gamma)\, \theta_2) \vdash_K^m (\hat{z}, v, (B, \gamma)\, \theta_2))$$
$$\vdash_K^{n-m} (z', e, \theta')$$

and

$$(z, v, \Phi((A, \gamma)\, \theta_2)) \vdash_{K'} (\hat{z}, v, \Phi((B, \gamma)\, \theta_2)) \vdash_{K'}^* (z'', e, \theta'')$$

with $z'' \in F$ follows. If $m > n$ then $v = e$ and

$$(z, e, (A, \gamma)\,\theta_2) \vdash_{\overline{K}} (z_1, e, (A, \gamma)(B, \gamma)\,\theta_2) \overset{m}{\vdash_{\overline{K}}} (\hat{z}, e, (B, \gamma)\,\theta_2).$$

Since K is F-preserving, $\hat{z} \in F$ holds. Thus we have $(z, e, \Phi((A, \gamma)\,\theta_2)) \vdash_{K'} (z'', e, \theta'')$ with $z'' = \hat{z} \in F$ and $\theta'' = \Phi((B, \gamma)\,\theta_2)$. This completes the induction.

Now let $w \in L(K)$, i.e., $(z_0, w, (A_0, g_0)) \overset{*}{\vdash}_K (z, e, \theta')$ with $z \in F$. Thus

$$(z_0, w, (A_0', g_0')) = (z_0, w, \Phi(A_0, g_0)) \overset{*}{\vdash_{K'}} (z'', e, \theta'') \qquad \text{with} \quad z'' \in F,$$

and therefore $w \in L(K')$.

To prove the converse inclusion, the following result is needed:

If $(z_0, w, (A_0', g_0')) \overset{n}{\vdash}_{K'} (z, v, \theta)$ with $\theta = \theta_1 \cdots \theta_s$, $\theta_i \in \Gamma_1' \times \Gamma_2'^*$, $s \geqslant 0$, $n \geqslant 0$, and $\theta_i = ((A_i, M_{i1}), (f_{i1}, M_{i2}) \cdots (f_{ir_i}, M_{i,r_i+1}))$ then $M_{ij} = \psi_K(f_{ij} \cdots f_{ir_i})$ for $j \in [1: r_i]$, $M_{i,r_i+1} = \varnothing$ and $r_i \geqslant 0$ for $i \in [1: s]$.

This fact, which is easily proved by induction on n, implies the following property of K': If K' starts in an initial configuration, the transition function assures that the second component of the topmost pushdown list symbol always has the desired value.

Let $\tau: (\Gamma_1' \times \Gamma_2'^*)^* \to (\Gamma_1 \times \Gamma_2^*)^*$ be the homomorphism defined by $\tau((A, M), (f_1, M_1) \cdots (f_r, M_r)) = (A, f_1 \cdots f_r)$. Now it will be proved that $L(K') \subseteq L(K)$. To this end it will be shown by induction on n:

If $(z_0, w, (A_0', g_0')) \overset{n}{\vdash}_{K'} (z, v, \theta)$ then $(z_0, w, (A_0, g_0)) \overset{*}{\vdash}_K (z, v, \tau(\theta))$. The assertion holds for $n = 0$. Assume the assertion holds for all $k \leqslant n$. Consider

$$(z_0, w, (A_0', g_0')) \overset{n}{\vdash_{K'}} (z, xv, \theta_1\theta) \vdash_{\overline{K'}} (z', v, \theta') \qquad \text{with} \quad x \in X \cup \{e\}$$

and

$$\theta_1 = ((A, M_1), (f_1, M_2) \cdots (f_r, M_{r+1})), \qquad r \geqslant 0.$$

We have $M_1 = \psi_K(f_1 \cdots f_r)$ and from the induction hypothesis $(z_0, w, (A_0, g_0)) \overset{*}{\vdash}_K (z, xv, \tau(\theta_1\theta))$ follows.

Now we have to consider four cases, since K' is normalized.

(1) $\delta'(z, x, ((A, M_1), (f_1, M_2))) = (z', ((B, M_2), e))$. Then $\delta(z, x, (A, f_1)) = (z', (B, e))$ and therefore $(z, xv, \tau(\theta_1\theta)) \vdash_K (z', v, \tau(\theta'))$.

(2.1) $\delta'(z, e, ((A, M_1), (f_1, M_2))) = (z', ((B, M'), (g, M_1)(f_1, M_2)))$ with $M' = \phi_K(g, M_1)$. Then $\delta(z, e, (A, f_1)) = (z', (B, gf_1))$ and therefore $(z, xv, \tau(\theta_1\theta)) \vdash_K (z', v, \tau(\theta'))$.

(2.2) $\delta'(z, e, ((A, M_1), (f_1, M_2))) = (z', ((B, M_1), (f_1, M_2)))$. Then we have either $\delta(z, e, (A, f_1)) = (z', (B, f_1))$ and therefore $(z, xv, \tau(\theta_1\theta)) \vdash_K (z', v, \tau(\theta'))$, or $\delta(z, e, (A, f_1)) = (z_2, (A, f_1)(B, f_1))$ with $(z_2, A, z') \in M_1$ and therefore $(z, xv, \tau(\theta_1\theta)) \vdash_K (z_2, v, (A, f_1 \cdots f_r)(B, f_1 \cdots f_r)\,\tau(\theta))$. Since

$(z_2, A, z') \in M_1 = \psi_K(f_1 \cdots f_r)$ we have $(z_2, e, (A, f_1 \cdots f_r)) \overset{*}{\vdash}_K (z', e, e)$ and thus $(z_2, v, (A, f_1 \cdots f_r)(B, f_1 \cdots f_r)\tau(\theta)) \overset{*}{\vdash}_K (z', v, (B, f_1 \cdots f_r)\tau(\theta)) = (z', v, \tau(\theta'))$.

All configurations of these moves except the last have a pushdown list of length $\geqslant |\theta| + 2$.

(3) $\delta'(z, e, ((A, M_1), (f_1, M_2))) = (z', ((A, M_1), (f_1, M_2))((B, M_1), (f_1, M_2)))$. Then $\delta(z, e, (A, f_1)) = (z', (A, f_1)(B, f_1))$ and therefore $(z, xv, \tau(\theta_1\theta)) \vdash_K (z', v, \tau(\theta'))$.

(4) $\delta'(z, e, ((A, M_1), (f_1, M_2))) = (z', e)$. Then $\delta(z, e, (A, f_1)) = (z', e)$ and therefore $(z, xv, \tau(\theta_1\theta)) \vdash_K (z', v, \tau(\theta'))$.

Now suppose $w \in L(K')$, i.e., $(z_0, w, (A_0', g_0')) \overset{*}{\vdash}_{K'} (z', e, \theta)$ with $z' \in F$. Then we have $(z_0, w, (A_0, g_0)) \overset{*}{\vdash}_K (z', e, \tau(\theta))$, hence $w \in L(K)$.

It remains to show that K' is e-monotone. Assume that K' is not e-monotone. Then there exists a sequence of moves

$$\begin{aligned}(z_0, w, (A_0', g_0')) &\overset{*}{\underset{K'}{\vdash}} (z, v, ((A, M), \gamma')\theta)\\ &\underset{K'}{\vdash} (z_1, v, ((A, M), \gamma')((B, M), \gamma')\theta) \underset{K'}{\vdash} (z_2, v, \theta_2)\\ &\underset{K'}{\vdash} (z_3, v, \theta_3) \underset{K'}{\vdash} \cdots \underset{K'}{\vdash} (z_r, v, ((B, M), \gamma')\theta)\end{aligned}$$

with $|\theta_i| \geqslant |\theta| + 2$ for $i \in [2: r-1]$. Then with $\tau((A, M), \gamma') = (A, \gamma)$ we have

$$(z_0, w, (A_0, g_0)) \overset{*}{\underset{K}{\vdash}} (z, v, (A, \gamma)\,\tau(\theta)) \underset{K}{\vdash} (z_1, v, (A, \gamma)(B, \gamma)\,\tau(\theta)).$$

Furthermore there exists n_j, $j \in [1: r-1]$ such that

$$\begin{aligned}(z_1, v, (A, \gamma)(B, \gamma)\,\tau(\theta)) &\overset{n_1}{\underset{K}{\vdash}} (z_2, v, \tau(\theta_2)) \overset{n_2}{\underset{K}{\vdash}} (z_3, v, \tau(\theta_3))\\ &\cdots \overset{n_{r-1}}{\underset{K}{\vdash}} (z_r, v, (B, \gamma)\,\tau(\theta)),\end{aligned}$$

where all configurations of these moves, except of the last, have a pushdown list of length $\geqslant |\theta| + 2$. Hence $(z_1, A, z_r) \in \psi_K(\gamma) = M$, and this implies

$$\delta'(z, e, (A, M), (g, M')) = (z_r, ((B, M), (g, M'))) \qquad \text{with } \gamma' = (g, M')\,\hat{\gamma}.$$

This contradicts the fact

$$(z, v, ((A, M), \gamma')\theta) \underset{K'}{\vdash} (z_1, v, ((A, M), \gamma')((B, M), \gamma')\theta).$$

In analogy to the deterministic pushdown automata we introduce the concept of a looping configuration. Then we will show that it is decidable whether a configuration is a looping configuration.

DEFINITION 3.5. A configuration $(z, e, (A, \gamma))$ of a d-IPDA is called *looping* if for all $n \geqslant 1$ there exists a configuration (z_n, e, θ_n) with $(z, e, (A, \gamma)) \vdash^n (z_n, e, \theta_n)$.

THEOREM 3.4. *It is decidable whether a configuration* $(\hat{z}, e, (A, \gamma))$ *of a d-IPDA* $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$ *is looping.*

Proof. Define an IPDA $K' = (Z', \varnothing, \Gamma_1', \Gamma_2', \delta', z_0', A_0', e, F')$ in the following manner: $Z' = Z \cup \{z_0', z_f\}$, z_0', z_f are new states, $\Gamma_1' = \Gamma_1 \cup \{A_0'\}$, $\Gamma_2' = \Gamma_2 \cup \{\#\}$, $F' = \{z_f\}$, the transition function δ' is defined by: $\delta'(z_0', e, (A_0', e)) = \{(\hat{z}, (A, \gamma\#)(A_0', e))\}$, and for all $z \in Z$ and $B \in \Gamma_1$:

If $\delta(z, e, (B, e)) \neq \varnothing$ then

$$\delta'(z, e, (B, e)) = \delta(z, e, (B, e)).$$

If $\delta(z, e, (B, e)) = \varnothing$ then for all $g \in \Gamma_2$

$$\begin{array}{ll} \delta'(z, e, (B, g)) = \delta(z, e, (B, g)), & \text{if } \delta(z, e, (B, g)) \neq \varnothing \\ \text{and } \delta', (z\ e, (B, \#)) = \{(z_f, e)\} & \text{otherwise} \end{array}$$

For all $z \in Z$:

$$\delta'(z, e, (A_0', e)) = \{(z_f, e)\}.$$

(K' is a d-IPDA.) K' has the following property: $L(K') = \varnothing$ iff $(\hat{z}, e, (A, \gamma))$ is a looping configuration. Assume $L(K') = \{e\} \neq \varnothing$, then we have

$$(z_0', e, (A_0', e)) \vdash_{\overline{K'}} (\hat{z}, e, (A, \gamma\#)(A_0', e)) \vdash_{K'}^{*} (z', e, (B, \gamma')\, \theta') \vdash_{\overline{K'}} (z_f, e, \theta).$$

If $|\theta'| = 0$ then $(B, \gamma') = (A_0', e)$, hence in K we have: $(\hat{z}, e, (A, \gamma)) \vdash^{*}_{K} (z', e, e)$, i.e., $(\hat{z}, e, (A, \gamma))$ is not a looping configuration.

Now assume $|\theta'| > 0$. If $(B, \gamma') = (B, \#)$ then in K we have $(\hat{z}, e, (A, \gamma)) \vdash^{*}_{K} (z', e, (B, e)\, \theta')$ and $\delta(z', e, (B, e)) = \varnothing$. If $(B, \gamma') = (B, g\gamma''\#)$ with $g \in \Gamma_2$ then in K we have $(\hat{z}, e, (A, \gamma)) \vdash^{*}_{K} (z', e, (B, g\gamma'')\, \theta')$ and $\delta(z', e, (B, e)) = \delta(z', e, (B, g)) = \varnothing$. In both cases we can conclude that $(\hat{z}, e, (A, \gamma))$ is not a looping configuration.

Now, on the other hand, assume that $(\hat{z}, e, (A, \gamma))$ is not a looping configuration. Then there exists an $n \geqslant 0$ with $(\hat{z}, e, (A, \gamma)) \vdash^{n}_{K} (z', e, \theta)$ and

(1) $\theta = e$ or

(2) $\theta = (B, e)\, \theta'$ and $\delta(z', e, (B, e)) = \varnothing$ or

(3) $\theta = (B, g\gamma')\, \theta'$ with $g \in \Gamma_2$ and

$$\delta(z', e, (B, e)) = \delta(z, e, (B, g)) = \varnothing.$$

In case (1) we have $(z_0', e, (A_0', e)) \vdash_{K'} (\hat{z}, e, (A, \gamma\#)(A_0', e)) \vdash_{K'}^{n} (z', e, (A_0', e)) \vdash_{K'} (z_f, e, e)$.

In case (2) we have $(z_0', e, (A_0', e)) \vdash_{K'} (\hat{z}, e, (A, \gamma\#)(A_0', e)) \vdash_{K'}^{n} (z', e, (B, \#)\,\theta') \vdash_{K'} (z_f, e, \theta')$.

In case (3) we have $(z_0', e, (A_0', e)) \vdash_{K'} (\hat{z}, e, (A, \gamma\#)(A_0', e)) \vdash_{K'}^{n} (z', e, (B, g\gamma'\#)\,\theta') \vdash_{K'} (z_f, e, \theta')$.

In all three cases we can conclude $L(K') = \{e\}$.

Now construct an indexed grammar G with $L(G) = L(K')$. The emptiness problem for indexed grammars is decidable, see Aho (1968) and Maibaum (1978).

If an e-monotone d-IPDA reaches a configuration $(z, v, (A, \gamma)\theta)$, where $(z, e, (A, \gamma))$ is a looping configuration, then the length of the pushdown lists of all subsequent configurations are monotone increasing.

Definition 3.6. Let $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$ be a d-IPDA. A looping configuration $(z, e, (A, \gamma))$ of K is called *monotone* if $(z, e, (A, \gamma)) \vdash_K^* (z_1, e, \theta_1) \vdash_K (z_2, e, \theta_2)$ implies $|\theta_1| \leqslant |\theta_2|$. A looping configuration $(z, e, (A, \gamma))$ of K is called *F-reaching* if there exists an $n \geqslant 0$ with $(z, e, (A, \gamma)) \vdash_K^n (z', e, \theta)$ and $z' \in F$.

Now let $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$ be a d-IPDA. Set $\mathscr{T}_K = Z \times \Gamma_1 \times \{1, 2\}$ and $\beta_K : \Gamma_2^* \to \mathscr{P}(\mathscr{T}_K)$ with

$$\begin{aligned}\beta_K(\gamma) = {} & \{(z, A, 1) \mid (z, e, (A, \gamma)) \text{ is a monotone looping configuration of } K \text{ which is } F\text{-reaching}\} \cup \\ & \{(z, A, 2) \mid (z, e, (A, \gamma)) \text{ is a monotone looping configuration of } K \text{ which is not } F\text{-reaching}\}\end{aligned}$$

β_K has a property like ϕ_K expressed by the following Lemma.

Lemma 3.2. *If $\beta_K(\gamma_1) = \beta_K(\gamma_2)$ then $\beta_K(g\gamma_1) = \beta_K(g\gamma_2)$ for $g \in \Gamma_2$ and $\gamma_1, \gamma_2 \in \Gamma_2^*$.*

Proof. Let $(z, A, k) \in \beta_K(g\gamma_1)$, $k \in [1, 2]$. If for all $n \geqslant 0$ we have $(z, e, (A, g\gamma_1)) \vdash_K^n (z', e, (B, \alpha\gamma_1)\theta)$ with $|\alpha| \geqslant 1$, then $(z, e, (A, g\gamma_2)) \vdash_K^n (z', e, (B, \alpha\gamma_2)\sigma(\theta))$, where $\sigma: (\Gamma_1 \times \Gamma_2^*\gamma_1)^* \to (\Gamma_1 \times \Gamma_2^*\gamma_2)^*$ is the homomorphism defined by $\sigma(A, \alpha\gamma_1) = (A, \alpha\gamma_2)$. Hence $(z, A, k) \in \beta_K(g\gamma_2)$.

Otherwise choose the minimal m with $(z, e, (A, g\gamma_1)) \vdash_K^m (z', e, (B, \gamma_1)\theta)$. Thus $(z, e, (A, g\gamma_2)) \vdash_K^m (z', e, (B, \gamma_2)\sigma(\theta))$ holds and the fact that $(z', e, (B, \gamma_1))$ is a monotone looping configuration and hence $(z', B, k') \in \beta_K(\gamma_1) = \beta_K(\gamma_2)$ for $k' = 1$ or $k' = 2$ implies that $(z, e, (A, g\gamma_2))$ is a monotone looping configuration. Hence $(z, A, \bar{k}) \in \beta_K(g\gamma_2)$ with $\bar{k} \in [1: 2]$.

It is easy to show that $k = \bar{k}$, and hence $\beta_K(g\gamma_1) \subseteq \beta_K(g\gamma_2)$. The other inclusion is proved similarly.

Now define a function $\rho_K : \Gamma_2 \times \mathscr{P}(\mathscr{T}_K) \to \mathscr{P}(\mathscr{T}_K)$ in the following way:

$$\begin{aligned} \rho_K(g, T) &= \beta_K(g\gamma) && \text{if} \quad T = \beta_K(\gamma) \quad \text{with} \quad \gamma \in \Gamma_2^* \\ &= \varnothing && \text{otherwise.} \end{aligned}$$

From Lemma 3.2. follows that ρ_K is well defined.

Now we can state

THEOREM 3.5. *For each d-IPDA K there exists an equivalent, normalized, e-monotone, F-preserving d_0-IPDA, which halts on each input.*

In the following proof we will consider w.l.o.g. an e-monotone d_0-IPDA K and construct a d_0-IPDA K' which simulates K. The only exception is that K' halts whenever K reaches a configuration $(z, v, (A, \gamma)\theta)$, where $(z, e, (A, \gamma))$ is a looping configuration. Furthermore it is important to know whether K reaches a final state starting from $(z, v, (A, \gamma)\theta)$. The necessary information is contained in $\beta_K(\gamma)$. In analogy to the proof of Theorem 3.3. the pushdown list symbols and indices will be extended by a second component containing this information. The necessary modification of the transition function is possible according to properties of β_K stated in Lemma 3.2.

Proof. W.l.o.g. we can assume that $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$ is a normalized, F-preserving, and e-monotone d_0-IPDA (cf. Theorem 3.3.). Define $K' = (Z', X, \Gamma_1', \Gamma_2', \delta', z_0, A_0', g_0', F')$ with $Z' = Z \cup \{q_1, q_2\}$, q_1, q_2 are new states, $\Gamma_1' = \Gamma_1 \times \mathscr{P}(\mathscr{T}_K)$, $\Gamma_2' = \Gamma_2 \times \mathscr{P}(\mathscr{T}_K)$, $A_0' = (A_0, T_0)$ with $T_0 = \beta_K(g_0)$, $g_0' = (g_0, \varnothing)$, $F' = F \cup \{q_1\}$, and δ' is defined by the following four cases: For all $z \in Z$, $(A, T) \in \Gamma_1'$, $(g, T') \in \Gamma_2'$:

(1) If $\delta(z, x, (A, g)) = (z', (B, e))$ with $x \in X \cup \{e\}$ then $\delta'(z, x, ((A, T), (g, T'))) = (\tilde{z}, ((B, T'), e))$ with

$$\begin{aligned} \tilde{z} &= q_k && \text{if} \quad (z, A, k) \in T \\ &= z' && \text{otherwise.} \end{aligned}$$

(2) If $\delta(z, e, (A, g)) = (z', (B, g))$ then $\delta'(z, e, ((A, T), (g, T'))) = (\tilde{z}, ((B, T), (g, T')))$ with $\tilde{z}$ as in (1). If $\delta(z, e, (A, g)) = (z', (B, fg))$ with $f \in \Gamma_2$ then $\delta'(z, e, ((A, T), (g, T'))) = (\tilde{z}, ((B, T''), (f, T)(g, T')))$ with $T'' = \rho_K(f, T)$ and $\tilde{z}$ as in (1).

(3) If $\delta(z, e, (A, g)) = (z', (A, g)(B, g))$ then $\delta'(z, e, ((A, T), (g, T'))) = (\tilde{z}, ((A, T), (g, T'))((B, T), (g, T')))$ with $\tilde{z}$ as in (1).

(4) If $\delta(z, e, (A, g)) = (z', e)$ then $\delta'(z, e, ((A, T), (g, T'))) = (z', e)$.

K' is a normalized, F-preserving, and e-monotone d_0-IPDA.

Let η: $(\Gamma_1' \times \Gamma_2'^*)^* \to (\Gamma_1 \times \Gamma_2^*)^*$ be the homomorphism defined by

$$\eta((A, T), (g_1, T_1) \cdots (g_r, T_r)) = (A, g_1 \cdots g_r).$$

First we have:

If $(z_0, w, (A'_0, g'_0)) \vdash^*_{K'} (z, v, ((A, T), \gamma')\theta)$ then $T = \beta_K(\gamma)$, where $\eta((A, T), \gamma') = (A, \gamma)$.

Now it will be proved:

If $(z_0, w, (A_0, g_0)) \vdash^n_K (z, v, (A, \gamma)\theta) \vdash_K (z_1, v_1, \theta_1)$ with $n \geqslant 0$ and $(z, e, (A, \gamma))$ is not a looping configuration then $(z_0, w, (A'_0, g'_0)) \vdash^n_{K'} (z, v, ((A, T), \gamma')\theta') \vdash_{K'} (z_1, v_1, \theta'_1)$ with $\eta(((A, T), \gamma')\theta') = (A, \gamma)\theta$ and $\eta(\theta'_1) = \theta_1$. This assertion holds for $n = 0$.

Assume that the assertion holds for all $m \leqslant n$. Now let $(z_0, w, (A_0, g_0)) \vdash^n_K (z, v, (A, \gamma)\theta) \vdash_K (z_1, v_1, (A_1, \gamma_1)\theta_1) \vdash_K (z_2, v_2, \theta_2)$, where $(z_1, e, (A_1, \gamma_1))$ is not a looping configuration. Since K is e-monotone, $(z, e, (A, \gamma))$ cannot be a looping configuration. According to the induction hypothesis we have

$$(z_0, w, (A'_0, g'_0)) \vdash^n_{K'} (z, v, ((A, T), \gamma')\theta') \vdash_{K'} (z_1, v_1, ((A_1, T_1), \gamma'_1)\theta'_1)$$

where $\eta(((A_1, T_1), \gamma'_1)\theta'_1) = (A_1, \gamma_1)\theta_1$ and furthermore $(z_1, v_1, ((A_1, T_1), \gamma'_1)\theta'_1) \vdash_{K'} (z_2, v_2, \theta'_2)$ with $\eta(\theta'_2) = \theta_2$.

Now we will show that K' is equivalent to K.

(a) $L(K) \subseteq L(K')$. Let $w \in L(K)$. Then there exists an $n \geqslant 0$ with $(z_0, w, (A_0, g_0)) \vdash^n_K (z_1, e, \theta_1)$ and $z_1 \in F$. If $n = 0$, then $w \in L(K')$, too.

If $n \geqslant 1$ then $(z_0, w, (A_0, g_0)) \vdash^{n-1}_K (z, v, (A, \gamma)\theta) \vdash_K (z_1, e, \theta_1)$.

If $(z, e, (A, \gamma))$ is not a looping configuration then we have $w \in L(K')$. Assume $(z, e, (A, \gamma))$ is a looping configuration, then choose the minimal m with $(z_0, w, (A_0, g_0)) \vdash^m_K (z_2, e, (B, \gamma_2)\theta_2) \vdash^{n-m}_K (z_1, e, \theta_1)$ such that $(z_2, e, (B, \gamma_2))$ is a looping configuration.

If $m = 0$ then $w = e$ and we have, according to the definition of δ', $(z_0, w, (A'_0, g'_0)) \vdash_{K'} (q_1, e, \theta')$, hence $w \in L(K')$. If $m \geqslant 1$, then we have according to the assertion proved above and the definition of δ'

$$(z_0, w, (A'_0, g'_0)) \vdash^m_{K'} (z_2, e, ((B, T), \gamma'_2)\theta'_2) \vdash_{K'} (q_1, e, \theta''),$$

hence $w \in L(K')$.

(b) $L(K') \subseteq L(K)$. With an easy induction one can show that $(z_0, w, (A'_0, g'_0)) \vdash^n_{K'} (z, v, \theta')$ with $z \in Z$ and $n \geqslant 0$ implies $(z_0, w, (A_0, g_0)) \vdash^n_K (z, v, \theta)$, where $\eta(\theta') = \theta$.

Now let $w \in L(K')$. Then there exists an $n \geqslant 0$ with $(z_0, w, (A'_0, g'_0)) \vdash^n_{K'} (z, e, \theta')$ with $z \in F' = F \cup \{q_1\}$. If $z \in F$ then $w \in L(K)$ follows.

Assume $z = q_1$. Then we have

$$(z_0, w, (A'_0, g'_0)) \vdash^{n-1}_{K'} (z_1, e, ((A, T), \gamma')\theta'_1) \vdash_{K'} (q_1, e, \theta')$$

and hence $(z_1, A, 1) \in T$. Then, according to K, we have $(z_0, w, (A_0, g_0)) \vdash_K^{n-1} (z_1, e, (A, \gamma)\theta_1) \vdash_K^* (\bar{z}, e, \bar{\theta})$ with $\eta(((A, T), \gamma')\theta_1') = (A, \gamma)\theta_1$ and $\bar{z} \in F$. Hence we have $w \in L(K)$.

It remains to show that K' halts on each input. Assume there exists a $v \in X^*$, such that there exists an $m \geqslant 0$ and for all $n \geqslant 1$ there exist configurations (z_n, e, θ_n') with $(z_0, v, (A_0', g_0')) \vdash_{K'}^m (z_1, e, \theta_1') \vdash_{K'}^{n-1} (z_n, e, \theta_n')$. Assume that m is minimal with this property, i.e., for all $s \in [0: m-1](z_0, v, (A_0', g_0')) \vdash_{K'}^s (\bar{z}_s, \bar{v}_s, \bar{\theta}_s)$ implies $|\bar{v}_s| \geqslant 1$. For all $n \geqslant 1$ we have $z_n \in Z$ and $|\theta_n'| \geqslant 1$. Since K' is e-monotone and the pushdown list can be shortened only a finite number of times, there exists an n_0 which is minimal with the property that for all $n \geqslant n_0$ the inequality $|\theta_{n_0}'| \leqslant |\theta_n'|$ holds.

Let $\theta_{n_0}' = ((A, T), \gamma')\theta'$ with $\eta((A, T), \gamma') = (A, \gamma)$. According to K we have for all $n \geqslant n_0$:

$$(z_0, v, (A_0, g_0)) \xrightarrow[K]{m+n_0-1} (z_{n_0}, e, (A, \gamma)\eta(\theta')) \xrightarrow[K]{n-n_0} (z_n, e, \eta(\theta_n'))$$

with $|(A, \gamma)\eta(\theta')| \leqslant |\eta(\theta_n')|$. Since K is e-monotone, the sequence $(|\eta(\theta_n')|)_{n \geqslant n_0}$ is monotone increasing. Hence $(z_{n_0}, e, (A, \gamma))$ is a monotone looping configuration and therefore $(z_{n_0}, A, k) \in T$ for $k = 1$ or $k = 2$. Then, by definition of δ', $z_{n_0+1} = q_k$, which is a contradiction to $z_{n_0+1} \in Z$.

Now we are in the position to prove our main result, that the deterministic indexed languages are closed under complementation.

THEOREM 3.6. *Let $L = L(K)$ for a d-IPDA $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$. Then there exists a d-IPDA $\bar{K}$ with $L(\bar{K}) = \bar{L} = X^* \backslash L$.*

Proof. W.l.o.g. let K be an F-preserving d_0-IPDA which halts on each input. Since $X = \varnothing$ is trivial, we assume $X \neq \varnothing$. First we construct an equivalent d_0-IPDA K' with the property: Each configuration which can be reached from an initial configuration has a nonempty pushdown list and all index lists are nonempty. $K' = (Z', X, \Gamma_1', \Gamma_2', \delta', q_0, \$, \#, F)$ is defined by $Z' = Z \cup \{q_0\}$, q_0 is a new state, $\Gamma_1' = \Gamma_1 \cup \{\$\}$, $\Gamma_2' = \Gamma_2 \cup \{\#\}$ and δ' is declared as follows: $\delta'(q_0, e, (\$, \#)) = (z_0, (A_0, g_0\#)(\$, \#))$ and for all $(z, x, (A, g)) \in Z \times (X \cup \{e\}) \times \Gamma_1 \times \Gamma_2$ set $\delta'(z, x, (A, g)) = \delta(z, x, (A, g))$.

K' is an F-preserving d_0-IPDA, which is equivalent to K and halts on each input.

Now we modify K' such that in addition each input string is read completely. To this end set $\tilde{K} = (\tilde{Z}, X, \Gamma_1', \Gamma_2', \tilde{\delta}, q_0, \$, \#, F)$ with $\tilde{Z} = Z' \cup \{q\}$, q is a new state, and $\tilde{\delta}$ is defined by the following cases:

(1) For all $(z, x, (A, g)) \in Z' \times (X \cup \{e\}) \times \Gamma_1' \times \Gamma_2$ with $\delta'(z, x, (A, g)) \neq \varnothing$ set $\tilde{\delta}(z, x, (A, g)) = \delta'(z, x, (A, g))$

(2) For all $z \in Z'$, $A \in \Gamma_1'$, $g \in \Gamma_2'$ with $\delta'(z, e, (A, g)) = \varnothing$ set for all $x \in X$ with $\delta'(z, x, (A, g)) = \varnothing : \tilde{\delta}(z, x, (A, g)) = (q, (A, g))$

(3) For all $x \in X$, $A \in \Gamma_1'$, $g \in \Gamma_2'$ set $\tilde{\delta}(q, x, (A, g)) = (q, (A, g))$.

$\tilde{K}$ is an F-preserving d_0-IPDA which is equivalent to K'.

With the aid of $\tilde{K}$ we can define a d_0-IPDA. $\bar{K} = (\bar{Z}, X, \Gamma_1', \Gamma_2', \bar{\delta}, q_0, \$, \#, \bar{F})$ with $L(\bar{K}) = \bar{L}$. Set $\bar{Z} = \tilde{Z} \cup \bar{F}$, where $\bar{F} = \{\bar{z} \mid z \in \tilde{Z} \backslash F\}$, and $\bar{\delta}$ is defined by:

For all $z \in F$: For all $A \in \Gamma_1'$, $x \in X \cup \{e\}$, $g \in \Gamma_2'$ set $\bar{\delta}(z, x, (A, g)) = \tilde{\delta}(z, x, (A, g))$.

For all $z \in \tilde{Z} \backslash F$: For all $A \in \Gamma_1'$, $x \in X$, $g \in \Gamma_2'$ with $\tilde{\delta}(z, x, (A, g)) \neq \varnothing$ set $\bar{\delta}(z, e, (A, g)) = (\bar{z}, (A, g))$ and $\bar{\delta}(\bar{z}, x, (A, g)) = \tilde{\delta}(z, x, (A, g))$.

For all $z \in \tilde{Z} \backslash F$: For all $A \in \Gamma_1'$, $g \in \Gamma_2'$ set $\bar{\delta}(z, e, (A, g)) = \tilde{\delta}(z, e, (A, g))$.

$\bar{K}$ is a d_0-IPDA with the property that each configuration which can be reached from an initial configuration has a nonempty pushdown list and all index lists are nonempty. Now we will prove: $L(\bar{K}) = \overline{L(\tilde{K})} = \overline{L(K)}$.

Let $w \notin L(\tilde{K})$. Since $\tilde{K}$ reads each input, is F-preserving and halts on each input, we have $(q_0, w, (\$, \#)) \overset{*}{\vdash}_{\tilde{K}} (z, e, (A, g\gamma)\theta)$ with $z \notin F$ and $\tilde{\delta}(z, e, (A, g)) = \varnothing$. $\tilde{K}$ reads for all $x \in X$ the word wx, hence $\tilde{\delta}(z, x, (A, g)) \neq \varnothing$ for all $x \in X$. Thus we have according to $\bar{K}$: $(q_0, w, (\$, \#)) \overset{*}{\vdash}_{\bar{K}} (z, e, (A, g\gamma)\theta) \vdash_{\bar{K}} (\bar{z}, e, (A, g\gamma)\theta)$, which implies $w \in L(\bar{K})$.

On the other hand, let $w \in L(\bar{K})$, then $(q_0, w, (\$, \#)) \overset{*}{\vdash}_{\bar{K}} (z, e, (A, g\gamma)\theta) \vdash_{\bar{K}} (\bar{z}, e, (A, g\gamma)\theta)$ with $z \notin F$. Choose n_0 minimal with $(q_0, w, (\$, \#)) \vdash_{\bar{K}}^{n_0} (z', e, \theta') \overset{*}{\vdash}_{\bar{K}} (z, e, (A, g\gamma)\theta) \vdash_{\bar{K}} (\bar{z}, e, (A, g\gamma)\theta)$, i.e., for all $s \in [0: n_0 - 1]$ $(q_0, w, (\$, \#)) \vdash_{\bar{K}}^{s} (z'', v, \theta'')$ implies $|v| > 0$. Then we have according to $\tilde{K}$ $(q_0, w, (\$, \#)) \vdash_{\tilde{K}}^{n_0-|w|} (z', e, \theta') \overset{*}{\vdash}_{\tilde{K}} (z, e, (A, g\gamma)\theta)$ with $\tilde{\delta}(z, e, (A, g)) = \varnothing$. Since $\tilde{K}$ is F-preserving and $z \notin F$, we have $z' \notin F$ and furthermore all states reached after z' are not in F. Hence $w \notin L(K)$.

4. Conclusions

Concerning the hierarchical embedding of the DIL's one can state

Theorem 4.1. (1) *Each deterministic context-free language is a DIL.*

(2) *There are inherently ambiguous context-free languages which are DIL's.*

(3) *There are indexed languages which are not DIL's.*

Proof. (1) Extend a deterministic pushdown automaton by a one-element set of indices.

(2) In the following example we will specify a d-IPDA which accepts the language $L = \{a^i b^j c^k \mid i, j, k \geqslant 1, i = j \text{ or } j = k\}$. This language is an inherently ambiguous context-free language (cf. Maurer, 1969).

(3) The indexed languages are not closed under complementation (Aho, 1968) and with Theorem 3.6. the assertion follows.

EXAMPLE 4.1. Set $K = (Z, X, \Gamma_1, \Gamma_2, \delta, z_0, A_0, g_0, F)$ with $Z = \{z_0, z_1, z_2, z_3, z_4\}$, $X = \{a, b, c\}$, $\Gamma_1 = \{A_0, A\}$, $\Gamma_2 = \{g_0, g\}$, $F = \{z_4\}$, and define δ by

$$\begin{aligned}
\delta(z_0, a, (A_0, e)) &= (z_0, (A_0, g)) \\
\delta(z_0, b, (A_0, g)) &= (z_1, (A, e)(A_0, g)) \\
\delta(z_1, b, (A, g)) &= (z_1, (A, e)(A, g)) \\
\delta(z_1, b, (A, g_0)) &= (z_2, (A, g_0)(A, g_0)) \\
\delta(z_1, c, (A, g_0)) &= (z_4, (A, g_0)) \\
\delta(z_1, c, (A, g)) &= (z_3, e) \\
\delta(z_2, b, (A, g_0)) &= (z_2, (A, g_0)(A, g_0)) \\
\delta(z_2, c, (A, e)) &= (z_3, e) \\
\delta(z_3, c, (A, e)) &= (z_3, e) \\
\delta(z_3, e, (A_0, e)) &= (z_4, e) \\
\delta(z_4, c, (A, g_0)) &= (z_4, (A, g_0)).
\end{aligned}$$

We have $L(K) = \{a^i b^j c^k \mid i, j, k \geqslant 1, i = j \text{ or } j = k\}$.

Consider, for example,

$$\begin{aligned}
(z_0, a^i b^i c^k, (A_0, g_0)) &\overset{i}{\longmapsto} (z_0, b^i c^k, (A_0, g^i g_0)) \\
&\longmapsto (z_1, b^{i-1} c^k, (A, g^{i-1} g_0)(A_0, g^i g_0)) \\
&\overset{i-1}{\longmapsto} (z_1, c^k, (A, g_0)(A, g g_0) \cdots (A_0, g^i g_0)) \\
&\longmapsto (z_4, c^{k-1}, (A, g_0)(A, g g_0) \cdots (A_0, g^i g_0)) \\
&\overset{k-1}{\longmapsto} (z_4, e, (A, g_0)(A, g g_0) \cdots (A_0, g^i g_0)).
\end{aligned}$$

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